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ORIGINAL ARTICLE

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New models for locating a moving service facility

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Abstract In this paper we analyze a new location problem which is a generalization of the well-known single facility location model. This extension consists of introducing a general objective function and replacing fixed locations by trajectories. We prove that the problem is well-stated and solvable. A Weiszfeld type algorithm is proposed to solve this generalized dynamic single facility location problem on L^p spaces of functions, with $p \in (1, 2]$. We prove global convergence of our algorithm once we have assumed that the set of demand functions and the initial step function belong to a subspace of L^p called Sobolev space. Finally, examples are included illustrating the application of the model to generalized regression analysis and the convergence of the proposed algorithm. The examples also show that the pointwise extension of the algorithm does not have to converge to an optimal solution of the considered problem while the proposed algorithm does.

Keywords Location · Weber Problem · Hyperbolic approximation

1 Introduction

In a location problem we are given the position of a number of demand facilities and the goal is to locate one or several service facilities to cover the demand in an optimal way. The objective function to be optimized depends of the nature of the problem although the most common notions are the minimization of the weighted sum or the maximum distances. The standard single facility location problem assumes that

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the position of the demand facilities is fixed “a priori” (Drezner 1995; Drezner and Hamacher 2002; Francis et al. 1992). Nevertheless, depending on what demand position means in each problem, it is clear that in some situations the positions may vary throughout the planning horizon. These models adapt better than the classical (static) ones to situations with seasonal demand, i.e., models where the demand changes depending on the season of the year. Scanning the literature of location analysis we can find several references dealing with problems where the initial definition of a location problem (position of the demand facilities or demand intensity) changes along the time.

Abdel-Malek (1985) considered the problem of optimal positioning of a service among moving existing ones, minimizing the weighted sum of distances in an interval. Drezner and Wesolowsky (1991), introduced a modification of the Weber problem allowing the location of the demand points to change a finite number of times. Compared with original approaches, the main improvement of that formulation is that it provides a better fit to real applications where data are time dependent, as for instance seasonal demand. Following these approaches, Puerto and Rodríguez-Chía (1999), deal with an extension of the previous model called the Dynamic Weber Problem.

In this paper, we present a general version of the Dynamic Single Facility Location problem where the objective function is an increasing, continuously differentiable function rather than the sum function; and that includes as particular instances classical, (Brimberg and Love 1993; Chandrasekaran and Tamir 1990; Frenk et al. 1994; Morris and Verdini 1979; Wesolowsky 1993) as well as dynamic models, (Abdel-Malek 1985; Drezner and Wesolowsky 1991; Puerto and Rodríguez-Chía 1999) previously studied in the literature. The goals in this paper are the following: (1) to propose a general formulation for single facility location problems with moving service facilities; (2) to develop an algorithm to solve this kind of problems; and (3) to prove global convergence for any sequence generated by this algorithm and for all $p \in (1, 2]$.

Although there exist algorithms in the literature that solve the static version of the problem, as we will show in the paper, the optimal solutions of the dynamic version of the problem do not have to coincide with the solution obtained when solving the problem optimally for each time epoch in a specific time interval, which we call the pointwise solution [see Brimberg and Love (1993), Brimberg et al. (1998), Cánovas et al. (2002), Frenk et al. (1994), Üster and Love (2000), Vardi and Zhang (2001) for details of Weiszfeld's algorithm in $\mathbb{R}^m$]. This counterintuitive performance reinforces the necessity of our analysis which may be mathematically explained by the different expressions of the iterates of each algorithm (compare (14) and (15) in section 4) and it is due to the different topological structure induced by the norm in the considered spaces.

Apart from the theoretical interest and the fact that this model fits better to problem with seasonal demand, there exists a clear application of this model in generalized regression analysis. Since the reader may not be familiar with this field we describe it in some detail. The standard problem in this field looks for the parameters defining a particular functional form which best fit a given set of data. It is well-known that in the least square model (l_2^2-regression): given

$A = \{(t_1, a_1), \ldots, (t_n, a_n)\}$ the data set and $f(t) = \lambda t + \mu$ the functional form; the problem consists of finding the minimum of

$$\min_{\lambda,\mu} \sum_{i=1}^{n} (|a_i - f(t_i)|)^2.$$

It is clear that this problem looks for the line minimizing the sum of the quadratic errors of the estimated data (error function). Everybody knows that the solution of this problem can be obtained using the normal equations. Apart from this very simple model there are many other generalizations of the least square regression model; some considering different measures: absolute deviation regression (l_1-regression), maximum deviation regression (l_∞-regression), ... and also, allowing different functional forms for f (Rousseeuw and Yohai 1984; Rousseeuw 1987).

Assume now that we observe a finite number of continuous time experiments in equal length periods of time. For instance, the trajectory of a solid sinking in a fluid or a continuous time demand function. Then, we are given a set $A = \{a_1(t), \ldots, a_n(t)\}$ of trajectories for $t \in [0, T]$. The regression problem for a generic error function g is:

$$\min_{x \in \{\text{space of trajectories}\}} g(\|x(t) - a_1(t)\|, \ldots, \|x(t) - a_n(t)\|) \tag{1}$$

being $\| \cdot \|$ a norm in the space of functions where x belongs to. Essentially, this is a generalized regression model with continuous type data set and no hypothesis on the form of the final regressor. This is exactly the problem that we will solve in this paper.

This problem appears in forecasting the intensity of electricity necessary to cover customers demand. The daily instantaneous demand of electricity is clearly a time dependent process for which electricity companies have repeated records (sample paths). Each one of these records is a function which gives the intensity of electricity required in each instant of time. Deviation from the actual demand has an economic cost: excesses are lost and shortages lead to uncovered demand or extra production cost. Thus, it is clear the need of accurate estimates of the instantaneous demand. This means, looking for an estimation of the demand which minimizes the overall deviations with respect to the recorded demands throughout the planning horizon. Since in this model the data are functions of time and no a priori shapes for the demand can be foreseen, we must apply the generalized regression described previously.

In addition, the model in (2) and the proposed algorithm can be used to solve location problems and to perform non-parametric estimation of the mean, median or more complex functions of stochastic processes or time series. Indeed, as well as the classical Weber problem allows us to compute sample 1-principal points of random variables (Flury 1990), this new approach permits to compute 1-principal functions of stochastic processes. Finally, it should be noted that the dynamic approach adapts better than the static one in modeling certain location situations. For instance, the location of a trajectory of a moving service facility with respect to a set of fixed routes or corridors.

The paper is organized as follows. Section 2 introduces the model and proves some preliminary results. Section 3 studies conditions to ensure existence and uniqueness of optimal solutions. Section 4 develops the proposed algorithm. In section 5 convergence properties are studied showing that the algorithm converges to an optimal solution of the considered problem. This section also includes an application to generalized regression analysis and an illustrative example showing the different behavior of the pointwise and the proposed solution approaches. Finally, section 6 contains the conclusions of the paper.

2 Model formulation and previous results

Let us consider the normed space $X_p = L^p(I, \mathbb{R}^m)$ equipped with the norm $||x||_p = \left(\int_I \sum_{k=1}^{m} |x_k(t)|^p \, dt\right)^{\frac{1}{p}}$ where I is a bounded interval. Given a finite set $A \subset X_p$, whose elements are called demand functions, and an increasing continuously differentiable function $g: S \longrightarrow \mathbb{R}$ with $\mathbb{R}_+^{|A|} \subset S$ we consider the following optimization problem

$$\inf_{x \in X_p} f_0(x) := g(d(x)) \tag{2}$$

where $d(x) = \left(||x - a||_p\right)_{a \in A}$.

Notice that this formulation includes the classical and well-known Weber problem on finite dimensional spaces, (Brimberg and Love 1993; Frenk et al. 1994), and the dynamic Weber problem, (Puerto and Rodríguez-Chía 1999), if we take the function g as the sum of the components of the distance vector $d(x)$.

The nondifferentiability of the objective function f_0 at the demand functions leads us to consider an alternative optimization problem similar to the one suggested by Eyster et al. (1973). It consists of replacing each vector $v = (v_1, \ldots, v_m) \in X_p$ by $\xi_\varepsilon(v)(t) = \left(\xi_{\varepsilon,1}(v)(t), \ldots, \xi_{\varepsilon,m}(v)(t)\right)$ being $\xi_{\varepsilon,k}(v)(t) = (v_k(t)^2 + \varepsilon^2 \chi_I(t))^{\frac{1}{2}}$ with $k = 1, \ldots, m$, $\varepsilon > 0$ and $\chi_I : \mathbb{R} \longrightarrow \mathbb{R}$ the function defined as

$$\chi_I(t) = \begin{cases} 1 & \text{if } t \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

This is the so-called hyperbolic approximation and it is known that Problem (2) is uniformly approximated by

$$\inf_{x \in X_p} f_\varepsilon(x) := g(d_\varepsilon(x)) \tag{4}$$

where $d_\varepsilon(x) = \left(||\xi_\varepsilon(x - a)||_p\right)_{a \in A}$.

We will use Problem (4) to get an optimal solution of Problem (2) by means of an iterative scheme. Since the objective function f_ε is differentiable an optimal solution x_ε^* for (4) can be obtained using the necessary condition $\nabla f_\varepsilon(x_\varepsilon^*) = 0$.

Once the problem has been formulated, our first step is to obtain sufficient conditions which ensure that the Problems (2) and (4) are well-stated and their set of optimal solutions are included in the considered space.

The first result in this section is a localization theorem which gives an upper bound of the norm of the optimal solutions of (2) and (4).

Lemma 2.1 *If x^* is an optimal solution of Problem (2) or (4) then*

$$||x^*||_p \leq 2 \max_{a \in A} ||a||_p + 1.$$

Proof Since we are interested in ε small enough, we can consider, without loss of generality, Problem (4) with $\varepsilon < \min \left\{ 1, \frac{1}{m(I)^{\frac{1}{p}}} \right\}$, being $m(I)$ the Lebesgue measure of I. Since g is an increasing function of its arguments, if x^* is an optimal solution there exists no $y \in X_p$ such that $||\xi_\varepsilon(y-a)||_p < ||\xi_\varepsilon(x^*-a)||_p \quad \forall a \in A$. Then, $\forall y \in X_p \quad \exists a \in A$, such that $||\xi_\varepsilon(x^*-a)||_p \leq ||\xi_\varepsilon(y-a)||_p$. Thus, applying for $y = 0$

$$||\xi_\varepsilon(x^* - a)||_p \leq ||\xi_\varepsilon(a)||_p, \quad \text{for some} \quad a \in A.$$

Hence,

$$||x^*||_p - ||a||_p \leq ||\xi_\varepsilon(x^* - a)||_p \leq ||a||_p + \varepsilon m(I)^{\frac{1}{p}}, \quad \forall \varepsilon < \min \left\{ 1, \frac{1}{m(I)^{\frac{1}{p}}} \right\}.$$

So that,

$$||x^*||_p \leq 2 \max_{a \in A} ||a||_p + 1.$$

Hence, any optimal solution of Problem (4) is included in the set

$$C_p = \{y \in X_p \ : \ ||y||_p \leq 2 \max_{a \in A} ||a||_p + 1\}. \tag{5}$$

Notice that the same proof holds for Problem (2) if we take $\varepsilon = 0$. □

Lemma 2.1 allows us to prove a theorem which establishes the existence of optimal solutions for Problems (2) and (4).

Theorem 2.1 *Problems* (2) *and* (4) *have optimal solutions in $L^p(I, \mathbb{R}^m)$ for any $p \in (1, +\infty)$ and we can therefore replace* "inf" *by* "min" *in the statements of these problems.*

Proof The functions f and f_ε are continuous convex functions defined on X_p whose set of optimal solutions are included in the set C_p (introduced in (5)). Since C_p is a bounded, closed, convex set, Proposition 38.12 in Zeidler (1985) ensures that both problems have optimal solutions. □

Since f_ε is differentiable, any optimal solution of (4) has to belong to the set

$$\Gamma_{f_\varepsilon} = \{x \in X_p \ : \ \nabla f_\varepsilon(x) = 0\}. \tag{6}$$

In what follows, we look for a sufficient condition of uniqueness of Problem (4). In order to develop such a condition we will prove a previous lemma.

Lemma 2.2 *The function $||\xi_\varepsilon(y - a)||_p$ is strictly convex for all $a \in A$ and for any $p \in (1, +\infty)$.*

Proof Indeed, $\|\cdot\|_p$ is a strictly convex function for any $p \in (1, +\infty)$. Besides, $\xi_\varepsilon(\cdot - a)$ is a vector whose components are convex functions. Therefore, the composed function $\|\xi_\varepsilon(\cdot - a)\|_p$ is convex. In order to prove the strict convexity, it suffices to prove that the components of $\xi_\varepsilon(\cdot - a)$ are strictly convex functions. Indeed, for each $t \in I$ and any $y \in X_p$

$$\xi_\varepsilon(y-a)(t) = \left(\left\|\Big(y_1(t) - a_1(t), \varepsilon\chi_I(t)\Big)\right\|_2, \dots, \left\|\Big((y_m(t) - a_m(t), \varepsilon\chi_I(t)\Big)\right\|_2\right).$$

where $\|\cdot\|_2$ is the l_2-norm in $\mathbb{R}^2$.

We will proceed by contradiction. Assume $\xi_\varepsilon(\cdot - a)$ is not strictly convex, then there exist $x, y \in X_p$ and $B \subseteq I$ ($m(B) > 0$) being $x(t) \neq y(t)$, for all $t \in B$, such that the following equation holds almost everywhere (a.e.), for all k,

$$\begin{aligned}&\left\|\Big(\theta x_k(t) + (1-\theta)y_k(t) - a_k(t), \varepsilon\chi_I(t)\Big)\right\|_2 \\ &\quad = \theta\left\|\Big(x_k(t) - a_k(t), \varepsilon\chi_I(t)\Big)\right\|_2 + (1-\theta)\left\|\Big(y_k(t) - a_k(t), \varepsilon\chi_I(t)\Big)\right\|_2\end{aligned}$$

The condition above implies that Minkowski's inequality is an equation, then it must exist $\lambda \in \mathbb{R}$ satisfying (Spivak 1970):

$$\Big(\theta(x_k(t) - a_k(t)), \theta\varepsilon\Big) = \lambda\Big((1-\theta)(y_k(t) - a_k(t)), (1-\theta)\varepsilon\Big), \quad \forall t \in I. \tag{7}$$

Therefore, we obtain from the equality of the second entries in both sides of (7), we have that

$$\lambda = \frac{\theta}{1-\theta},$$

and from the equality of the first entries in (7) that

$$x_k(t) - a_k(t) = y_k(t) - a_k(t) \quad \text{(a.e.) and for any } 1 \le k \le m,$$

what implies that

$$x(t) = y(t), \quad \text{(a.e.)}.$$

This is a contradiction. Thus, we get the thesis of this lemma. □

The uniqueness result is given by the following theorem.

Theorem 2.2 *If the function $g : S \longrightarrow \mathbb{R}$ is quasiconvex on $\mathbb{R}^m_+$ and increasing continuously differentiable then Γ_{f_ε}, defined in* (6)*, only contains the unique optimal solution of Problem* (4).

Proof Since we have proved in Lemma 2.2 that
$d_{\varepsilon,a}(x) := ||\xi_\varepsilon(x-a)||_p$ is strictly convex then

$$d_{\varepsilon,a}(x+h) - d_{\varepsilon,a}(x) > \langle \nabla d_{\varepsilon,a}(x), h\rangle, \quad \forall a \in A \quad \forall h \in X_p; \tag{8}$$

where $\langle.,.\rangle$ is defined in the Appendix (see (16)).

Let us consider $\bar{x} \in \Gamma_{f_\varepsilon}$, then the directional derivative of f_ε at $\bar{x}$ in the direction h, $f'_\varepsilon(\bar{x}, h)$, verifies

$$0 = f'_\varepsilon(\bar{x}, h) = \sum_{a \in A} \frac{\partial g}{\partial z_a}(d_\varepsilon(\bar{x}))\langle \nabla d_{\varepsilon,a}(\bar{x}), h\rangle, \quad \forall h \in X_p.$$

Using the inequality (8) and the fact that g is increasing,

$$0 < \sum_{a \in A} \frac{\partial g}{\partial z_a}(d_\varepsilon(\bar{x}))\Big(d_{\varepsilon,a}(\bar{x} + h) - d_{\varepsilon,a}(\bar{x})\Big), \quad \forall h \in X_p.$$

This implies by the quasiconvexity of g and Theorem 3.5.4 of Bazaraa and Shetty (1979) that

$$f_\varepsilon(\bar{x} + h) > f_\varepsilon(\bar{x}) \quad \forall h \in X_p$$

and so $\bar{x}$ is the unique optimal solution of Problem (4). □

Notice that the same result could have been obtained under less restrictive hypothesis, for instance, it would have been enough that g were non decreasing with at least one increasing component.

3 The Weiszfeld dynamic hyperbolic algorithm

The goal of this section is to develop an algorithm to solve Problem (4). This algorithm consists of adapting the hyperbolic approximation of the Weiszfeld algorithm to this dynamic problem. Based on the necessary condition $\nabla f_\varepsilon(x^*_\varepsilon) = 0$ for x^*_ε to be an optimal solution of (4) one can construct an iterative scheme similar to the Weiszfeld algorithm. We want to point out that in this case the necessary condition is also sufficient by Theorem 2.2 and can be written as:

$$\frac{\partial f_\varepsilon}{\partial x_k}(x; h) \;=\; 0 \quad k = 1, \ldots, m \quad \forall h \in X_p \quad p > 1$$

where $\frac{\partial f_\varepsilon}{\partial x}(x; h)$ stands for the Gateaux differential at x in the direction of h.

Equivalently, this derivative can be written for a particular k, $1 \le k \le m$, as

$$\sum_{a \in A} \frac{\partial g}{\partial z_a}(d_\varepsilon(x))||\xi_\varepsilon(x - a)||_p^{1-p} \left(\int_I \xi_{\varepsilon,k}(x-a)(t)^{p-2}(x_k(t)-a_k(t))h_k(t)\; dt\right) = 0$$

for all $p > 1$ and for all $h \in X_p$.

Using the completeness of X_p, the following expression holds,

$$\sum_{a \in A} \frac{\partial g}{\partial z_a}(d_\varepsilon(x))||\xi_\varepsilon(x - a)||_p^{1-p}\xi_{\varepsilon,k}(x - a)(t)^{p-2}(x_k(t) - a_k(t)) = 0$$
$$\forall p > 1 \quad \forall k = 1, \ldots, m.$$

Hence, we obtain an iterative process by means of the fixed point equation, $T_\varepsilon(x) = x$; where $T_\varepsilon(x) = \Big(T_{\varepsilon,1}(x), \ldots, T_{\varepsilon,m}(x)\Big)$ is given by

$$T_{\varepsilon,k}(x)(t) = \sum_{a\in A} \frac{\frac{\partial g}{\partial z_a}(d_\varepsilon(x))||\xi_\varepsilon(x-a)||_p^{1-p}\xi_{\varepsilon,k}(x-a)(t)^{p-2}}{\sum_{b\in A}\frac{\partial g}{\partial z_b}(d_\varepsilon(x))||\xi_\varepsilon(x-b)||_p^{1-p}\xi_{\varepsilon,k}(x-b)(t)^{p-2}} a_k(t) \quad \forall k = 1, \ldots, m. \tag{9}$$

The equation for T_ε is well-defined because, $\xi_{\varepsilon,k}(x-a)(t) > 0, \forall a \in A, \quad \forall k = 1, \ldots, m$ and for any $t \in I$. Besides, $||\xi_\varepsilon(x-a)||_p^{1-p} > 0$ and $\dfrac{\partial g}{\partial z_a}(d_\varepsilon(x)) > 0$ for any $a \in A$ because g is an increasing function. Therefore, if we consider the fixed-point map $T_\varepsilon(x) = x$ with $x \in X_p$, we get the following iterative scheme:

$$x_k^{q+1}(t) = \sum_{a\in A} \frac{\frac{\partial g}{\partial z_a}(d_\varepsilon(x^q))||\xi_\varepsilon(x^q-a)||_p^{1-p}\xi_{\varepsilon,k}(x^q-a)(t)^{p-2}}{\sum_{b\in A}\frac{\partial g}{\partial z_b}(d_\varepsilon(x^q))||\xi_\varepsilon(x^q-b)||_p^{1-p}\xi_{\varepsilon,k}(x^q-b)(t)^{p-2}} a_k(t) \quad \forall k = 1, 2, \ldots, m. \tag{10}$$

This scheme will be referred later as dynamic hyperbolic Weiszfeld algorithm. This iterative method has two important properties. The first, which establishes that the scheme gives descent, is an adaptation of Theorem 3.2 in Frenk et al. (1994) and its proof is included for the sake of completeness. In this proof we will use the following lemma whose proof is due to Beckenbach and Bellman (1967).

Lemma 3.1 *If* $a, b > 0$, $u < 1$ *and* $\frac{1}{u} + \frac{1}{v} = 1$ *then*

$$a^{\frac{1}{u}} b^{\frac{1}{v}} \geq \frac{a}{u} + \frac{b}{v}.$$

Proposition 3.1 *For any* $1 < p \leq 2$, $f_\varepsilon(T_\varepsilon(x)) < f_\varepsilon(x)$ *provided that* $\nabla f_\varepsilon(x) \neq 0$.

Proof Let $\varphi(z) = g(z_1^{\frac{1}{p}}, \ldots, z_{|A|}^{\frac{1}{p}})$. Since g is, by hypothesis, increasing continuously differentiable function and $z^{\frac{1}{p}}$ with $1 < p \leq 2$ is concave then $\varphi(\cdot)$ is quasiconcave.

Consider the function $h_k\colon X_p \longrightarrow X_p$ defined by

$$h_k(y)(t) = \sum_{a\in A} \frac{\partial\varphi}{\partial z_a}(d_\varepsilon(x)^p)\xi_{\varepsilon,k}(x-a)(t)^{p-2}\xi_{\varepsilon,k}(y-a)(t)^2.$$

First, we prove that

$$h_k(T(x))(t) \leq h_k(x)(t), \quad \forall t \in I,\ \forall k = 1, \ldots, m. \tag{11}$$

In order to do that, we develop the following expression:

$$\begin{aligned}
h_k(T(x))(t) - h_k(x)(t) &= \sum_{a\in A} \frac{\partial\varphi}{\partial z_a}(d_\varepsilon(x)^p)\xi_{\varepsilon,k}(x-a)(t)^{p-2} \\
&\quad \times\Big(\xi_{\varepsilon,k}(T(x)-a)(t)^2 - \xi_{\varepsilon,k}(x-a)(t)^2\Big) \\
&= \sum_{a\in A} \frac{\partial\varphi}{\partial z_a}(d_\varepsilon(x)^p)\xi_{\varepsilon,k}(x-a)(t)^{p-2} \\
&\quad \times\Big((T_k(x)(t)-a_k(t))^2 - (x_k(t)-a_k(t))^2\Big) \\
&= \sum_{a\in A} \frac{\partial\varphi}{\partial z_a}(d_\varepsilon(x)^p)\xi_{\varepsilon,k}(x-a)(t)^{p-2} \\
&\quad \times\Big((T_k(x)(t)-x_k(t))^2 + 2(T_k(x)(t) \\
&\quad -x_k(t))(x_k(t)-a_k(t))\Big) \\
&= (T_k(x)(t)-x_k(t)) \sum_{a\in A} \frac{\partial\varphi}{\partial z_a}(d_\varepsilon(x)^p)\xi_{\varepsilon,k}(x-a)(t)^{p-2} \\
&\quad \times\Big((T_k(x)(t)-x_k(t)) \\
&\quad +2(x_k(t)-a_k(t))\Big). \qquad (12)
\end{aligned}$$

Since $\varphi(z_1,\ldots,z_{|A|}) = g(z_1^{\frac{1}{p}},\ldots,z_{|A|}^{\frac{1}{p}})$ we have that

$$\frac{\partial\varphi}{\partial z_a}(z) = \frac{1}{p}\frac{\partial g}{\partial z_a}(z) z_a^{\frac{1}{p}-1}$$

and by the definition of $T_k(x)$ we also have that

$$\begin{aligned}
&\sum_{a\in A} \frac{\partial g}{\partial z_a}(d_\varepsilon(x))||\xi_\varepsilon(x-a)||_p^{1-p}\xi_{\varepsilon,k}(x-a)(t)^{p-2}a_k(t) \\
&\quad = T_k(x)(t)\sum_{a\in A} \frac{\partial g}{\partial z_a}(d_\varepsilon(x))||\xi_\varepsilon(x-a)||_p^{1-p}\xi_{\varepsilon,k}(x-a)(t)^{p-2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(12) &= -\frac{1}{p}(T_k(x)(t)-x_k(t))^2 \sum_{a\in A} \frac{\partial g}{\partial z_a}(d_\varepsilon(x))||\xi_\varepsilon(x-a)||_p^{1-p}\xi_{\varepsilon,k}(x-a)(t)^{p-2} \\
&= -\frac{1}{p}\nabla_k f_\varepsilon(x)(t)^2\Big(\sum_{a\in A} \frac{\partial g}{\partial z_a}(d_\varepsilon(x))||\xi_\varepsilon(x-a)||_p^{1-p}\xi_{\varepsilon,k}(x-a)(t)^{p-2}\Big)^{-1} \\
&\le 0.
\end{aligned}$$

Thus, we have proved the inequality (11). Now, we define the function $S_k : X_p \longrightarrow X_p$ given by

$$S_k(y) = \sum_{a\in A} \frac{\partial\varphi}{\partial z_a}(d_\varepsilon(x)^p)\xi_{\varepsilon,k}(y-a)^p.$$

Notice that $h_k(y)$ and $S_k(y)$ coincide at $y = x$. Moreover by Lemma 3.1, with $a(t) = \xi_{\varepsilon,k}(T(x) - a)(t)^p$, $b(t) = \xi_{\varepsilon,k}(x - a)(t)^p$ and $\frac{1}{v} = \frac{2}{p}$ we obtain that

$$h_k(T(x))(t) \geq \frac{2}{p} S_k(T_k(x))(t) + \left(1 - \frac{2}{p}\right) S_k(x)(t) \quad \forall t \in I.$$

Hence, using inequality (11) and $S_k(x) = h_k(x)$ it follows that $S_k(T(x))(t) \leq S_k(x)(t)$ for any $k = 1, \ldots, m$, with at least one strict inequality since $\nabla f_\varepsilon(x) \neq 0$. Therefore, we have that

$$\sum_{k=1}^{m} S_k(T(x))(t) < \sum_{k=1}^{m} S_k(x)(t).$$

and

$$\sum_{a \in A} \frac{\partial \varphi}{\partial z_a}(d_\varepsilon(x)^p)(||\xi_\varepsilon(T(x) - a)||_p^p - ||\xi_\varepsilon(x - a)||_p^p) < 0.$$

Finally, since φ is quasiconcave we can use Theorem 3.5.4. in Bazaraa and Shetty (1979) to obtain that

$$\varphi(d_\varepsilon(T(x)^p) \leq \varphi(d_\varepsilon(x)^p),$$

and hence

$$f_\varepsilon(T(x)) \leq f_\varepsilon(x). \qquad \square$$

The second property ensures that under mild hypotheses the whole sequence generated by the algorithm is included in the Sobolev space $W^{1,p}(I, \mathbb{R}^m)$, (see Definition 5.1 in the Appendix for a description of this space).

It is worth noting that $x^o \in W^{1,p}(I, \mathbb{R}^m)$ is not an actual restriction. In most cases, the data observed in continuous time experiments are differentiable functions in any bounded interval, (smooth trajectories), this implies that x^o belongs to the Sobolev space.

Lemma 3.2 *If every demand function $a \in A$ belongs to $W^{1,p}(I, \mathbb{R}^m)$ and x^o also belongs to $W^{1,p}(I, \mathbb{R}^m)$ then the sequence generated by Algorithm* (10) *is included in $W^{1,p}(I, \mathbb{R}^m)$.*

Proof The algorithm is defined as

$$T_k(x)(t) = \sum_{a \in A} \phi_{a,x}(t) a(t)$$

where

$$\phi_{a,x}(t) = \frac{\frac{\partial g}{\partial z_a}(d_\varepsilon(x))||\xi_\varepsilon(x - a)||^{1-p} \xi_\varepsilon(x - a)(t)^{p-2}}{\sum_{b \in A} \frac{\partial g}{\partial z_b}(d_\varepsilon(x))||\xi_\varepsilon(x - b)||^{1-p} \xi_\varepsilon(x - b)(t)^{p-2}}.$$

Thus, to show that the sequence generated by T is included in $W^{1,p}(I, \mathbb{R}^m)$, it suffices to prove that $T(x) \in X_p$ and its derivative belongs to X_p for any x in $W^{1,p}(I, \mathbb{R}^m)$.

Since $0 \leq \phi_{a,x}(t) \leq 1$ for all t then $T(x)$ is bounded by the function $\sum_{a\in A} |a|$, which belongs to X_p. Hence, $T(x) \in X_p \quad \forall x \in W^{1,p}(I, \mathbb{R}^m)$.

To prove that the derivative of T(x) with respect to t belongs to X_p, we have that

$$T'(x)(t) = \sum_{a\in A} \phi'_{a,x}(t)a(t) + \sum_{a\in A} \phi_{a,x}(t)a'(t)$$

Since $0 \leq \phi_{a,x}(t) \leq 1$ and $a \in W^{1,p}(I, \mathbb{R}^m)$ we have that $\sum_{a\in A} \phi_{a,x}(t)a'(t) \in X_p$. In order to prove that $\sum_{a\in A} \phi'_{a,x}(t)a(t) \in X_p$ we compute $\phi'_{a,x}$, before that, we introduce the following functions to simplify the notation.

$$\begin{aligned} h_{a,x}(t) &= (\xi_\varepsilon(x-a)(t))^{p-2} \\ D_{a,x} &= \frac{\partial g}{\partial z_a}(d_\varepsilon(x)||\xi_{\varepsilon,x}(x-a))||_p^{1-p} \\ r_x(t) &= \Big(\sum_{a\in A} D_{a,x}h_{a,x}(t)\Big)^2 \end{aligned}$$

Thus, we can write down $\phi'_{a,x}(t)$ in the following way:

$$\phi'_{a,x}(t) = \frac{D_{a,x}h'_{a,x}(t)\sum_{b\in A} D_{b,x}h_{b,x}(t) - D_{a,x}h_{a,x}(t)\sum_{b\in A} D_{b,x}h'_{b,x}(t)}{r_x(t)}.$$

Computing the derivative of $h_{a,x}(t)$ with respect to t we obtain

$$h'_{a,x}(t) = (p-2)(x(t)-a(t))(x'(t)-a'(t))(\xi_\varepsilon(x-a)(t))^{p-4}$$

hence, if we denote

$$q_{a,x}(t) = \frac{D_{a,x}h_{a,x}(t)\sum_{b\in A} D_{b,x}h_{b,x}(t)}{(\sum_{b\in A} D_{b,x}h_{b,x}(t))^2}$$

we have that

$$\begin{aligned} \phi'_{a,x}(t) = (p-2)q_{a,x}(t)\Big(&(x(t)-a(t))(x'(t)-a'(t))\xi_\varepsilon^{-2}(x-a)(t) - (x(t) \\ &-b(t))(x'(t)-b'(t))\xi_\varepsilon^{-2}(x-b)(t)\Big) \end{aligned}$$

Moreover, (1) $q_{a,x}(t) \leq 1 \quad \forall t \in I$, (2)$|(x(t)-a(t))\xi_\varepsilon^{-1}(x-a)(t)| \leq 1$ for any $t \in I$, $a \in A$ and (3) $\xi_\varepsilon^{-1}(x-b)(t) \leq \varepsilon^{-1} \quad \forall b \in A$. Therefore, we obtain the following inequality;

$$\|\phi'_{a,x}(t)\|_p \leq 2(2-p)\varepsilon^{-1} \max_{a\in A} \|x'-a'\|_p.$$

Since, x and a belong to $W^{1,p}(I, \mathbb{R}^m)$ that implies $(x'-a')$ belongs to X_p, that means that $\|x'-a'\|_p$ is bounded, therefore we obtain that $\phi'_{a,x} \in X_p$.

Hence, $\phi_{a,x} \in W^{1,p}(I, \mathbb{R}^m)$, and by Lemma 5.1 of the Appendix, one has that $\phi_{a,x}(t)a(t) \in W^{1,p}(I, \mathbb{R}^m)$ thus $\sum_{a\in A} \phi_{a,x}(t)a(t) \in W^{1,p}(I, \mathbb{R}^m)$, i.e., $T(x) \in W^{1,p}(I, \mathbb{R}^m)$. □

4 The convergence of the algorithm

In this section, we study the convergence of the proposed algorithm for the generalized dynamic Weber problem. We will prove the global convergence of this scheme for $p \in (1, 2]$. First of all, it should be noted that, by the proof of Lemma 3.2, the sequence generated by Algorithm (10), is bounded in $W^{1,p}(I, \mathbb{R}^m)$. Therefore, it contains a subsequence weakly convergent in $W^{1,p}(I, \mathbb{R}^m)$ (Brezis 1983). However, this result is not enough and we look for additional conditions which ensure the strong convergence of the sequence (see the Appendix for further details on the difference between weak and strong convergence).

Theorem 4.1 *If the function g is quasiconvex and increasing, continuously differentiable on* $\mathbb{R}^n_+$; *A and* x^o *verify the hypothesis of Lemma* 3.2 *then the sequence generated by Algorithm* (10), *for a given* ε, *strongly converges to an optimal solution of Problem* (4).

Proof The sequence given by the algorithm contains a weakly convergent subsequence. By Lemma 3.2 the whole sequence belongs to the Sobolev space $W^{1,p}(I, \mathbb{R}^m)$ so that by Lemma 5.1 assertion (2) (in the Appendix) it is also strongly convergent in X_p. Besides, we also know that under these hypotheses Proposition 3.1 ensures that the whole sequence is descent, provided that $\nabla f_\varepsilon(x^q) \neq 0$. Therefore, we can apply Zangwill's theorem (Bazaraa and Shetty 1979) to obtain that the considered subsequence strongly converges to a function $x^*_\varepsilon \in X_p$ verifying $\nabla f_\varepsilon(x^*_\varepsilon) = 0$. Finally, Theorem 2.2 ensures that x^*_ε is the optimal solution of Problem (4).

Once, we have proved that there exists a subsequence strongly convergent, we have to show that the whole sequence is strongly convergent. In order to do that, notice that the sequence contains a unique accumulation point, x^*_ε. Indeed, any subsequence is descent, bounded and included in $W^{1,p}(I, \mathbb{R}^m)$, then we apply again Zangwill's theorem to any subsequence and it converges to an element of Γ_{f_ε}. Hence, the whole sequence is convergent, because Γ_{f_ε} is a singleton. □

Once we have an algorithm which converges to the solution of Problem (4), the final part of this section is devoted to develop a method to get an optimal solution of Problem (2).

For any $\varepsilon > 0$ consider the problem

$$P_\varepsilon : \min_{x \in X_p} f_\varepsilon(x).$$

Let us denote by $x^*_{\varepsilon_n}$ the optimal solution of Problem (P_{ε_n}) and consider any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ which converges to 0.

Lemma 4.1 *If the sequence* $\{x^*_{\varepsilon_n}\}_{n\in\mathbb{N}}$ *is bounded in* $W^{1,p}(I, \mathbb{R}^m)$ *then it contains a convergent subsequence in the strong topology of* X_p.

Proof Since the sequence $\{x^*_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded, for all $n \geq 1$, hence we can extract a weakly convergent subsequence from it. Finally, we apply Lemma 5.1 assertion (2) (see the Appendix) and the result follows. □

Theorem 4.2 *If the sequence* $\{x^*_{\varepsilon_n}\}_{n\in\mathbb{N}}$ *converges strongly to* x^* *then* $x^* \in \arg\min f_0$.

Proof First of all, since the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is decreasing $\{f_{\varepsilon_n}(x)\}_{n\in\mathbb{N}}$ is also a decreasing sequence for any $x \in X_p$. Thus, applying Theorem 2.46 in Attouch (1984) we obtain that the sequence $\{f_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is epi-convergent in the strong topology of X_p.

In addition, since $\{f_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is a decreasing sequence applying Proposition 2.48. in Attouch (1984) we have that:

$$\lim_{n\to\infty} \inf_{x\in X_p} f_{\varepsilon_n}(x) = \inf_{x\in X_p} \lim_{n\to\infty} f_{\varepsilon_n}(x) = \inf_{x\in X_p} f_0(x). \tag{13}$$

Then, as $L^p(I, \mathbb{R}^m)$ is a first countable space, $\{f_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is epi-convergent, the sequence $\{x^*_{\varepsilon_n}\}_{n\in\mathbb{N}}$ contains a convergent subsequence, and using (13), we can apply Corollary 2.13 in the above mentioned reference to have that x^* belongs to the set $\arg\min_{x\in X_p} f_0(x)$. □

For practical purposes Theorem 4.2 requires knowledge of an optimal solution of Problem (4) for each objective function f_{ε_n} for all $n \in \mathbb{N}$. However, in order to solve each one of these problems we have to apply again an iterative algorithm. Therefore, although we can obtain approximate values for each $x_n \in \arg\min f_{\varepsilon_n}$, the exact expression may not be computed. This drawback can be avoided using a diagonal scheme as shown in the next algorithm. Let $T^k_{\varepsilon_n}(x)$ denote k applications of T_{ε_n} on x, where T_ε was defined in (9). This is to say, $T^k_{\varepsilon_n}(x) = T_{\varepsilon_n}(T_{\varepsilon_n} \overset{k-3}{\cdots} T_{\varepsilon_n}(x))$.

Theorem 4.3 *Let* $\{y_n\}_{n\in\mathbb{N}}$ *be a sequence defined by* $y_n := T^n_{\varepsilon_n}(y_{n-1})$ *for any* $n \in \mathbb{N}$ *and bounded in* $W^{1,p}(I, \mathbb{R}^m)$. *Then, any accumulation point of this sequence is an optimal solution of Problem* (2).

Proof By Lemma 4.1 the sequence $\{x^*_{\varepsilon_n}\}_{n\in\mathbb{N}}$ contains a subsequence strongly convergent to x^* with $x^* \in \arg\min f_0$. Let $\{x^*_{n_k}\}_{k\in\mathbb{N}}$ be such a sequence. Let us consider the subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ defined by $y_{n_k} := T^{n_k}_{\varepsilon_{n_k}}(y_{n_{k-1}})$. Therefore we have,

$$\|y_{n_k} - x^*\|_p \le \|T^{n_k}_{\varepsilon_{n_k}}(y_{n_{k-1}}) - x^*_{n_k}\|_p + \|x^*_{n_k} - x^*\|_p.$$

Now, for any $\varepsilon > 0$, there exists n_k such that by Theorem 4.1, $\|T^{n_k}_{\varepsilon_{n_k}}(y_{n_{k-1}}) - x^*_{n_k}\|_p < \dfrac{\varepsilon}{2}$ and by Theorem 4.2, $||x^*_{n_k} - x^*||_p < \frac{\varepsilon}{2}$. Therefore $||y_{n_k} - x^*||_p < \varepsilon$ and the result is proved. □

In what follows an example is included illustrating the use of Weiszfeld dynamic hyperbolic algorithm. Moreover, it shows that the pointwise application of classical Weiszfeld's algorithm does not work with the dynamic Weber problem. This fact makes our algorithm useful.

Notice that although, the difference between the two solutions seems to be counterintuitive it can be explained. The expressions (14) and (15) give the formulas of the pointwise hyperbolic Weiszfeld algorithm and the dynamic hyperbolic Weiszfeld algorithm at t:

$$x^{q+1}(t) = \sum_{a\in A} \frac{\frac{\partial g}{\partial z_a}(d_\varepsilon(x^q(t)))||\xi_\varepsilon(x^q(t) - a(t))||_p^{1-p}\xi_\varepsilon(x^q - a)(t)^{p-2}}{\sum_{b\in A} \frac{\partial g}{\partial z_b}(d_\varepsilon(x^q(t)))||\xi_\varepsilon(x^q(t) - b(t))||_p^{1-p}\xi_\varepsilon(x^q - b)(t)^{p-2}} a(t) \tag{14}$$

$$x^{q+1}(t) = \sum_{a\in A} \frac{\frac{\partial g}{\partial z_a}(d_\varepsilon(x^q))||\xi_\varepsilon(x^q-a)||_p^{1-p}\xi_\varepsilon(x^q-a)(t)^{p-2}}{\sum_{b\in A}\frac{\partial g}{\partial z_b}(d_\varepsilon(x^q))||\xi_\varepsilon(x^q-b)||_p^{1-p}\xi_\varepsilon(x^q-b)(t)^{p-2}}a(t) \quad (15)$$

The expression (14) does not depend on the norm in X_p of $\xi_\varepsilon(x^q - a)$ with $a \in A$, i.e., $\|\xi_\varepsilon(x^q - a)\|_p$. It depends on the l_p-norm in $\mathbb{R}^m$ of $\xi_\varepsilon(x^p - a)(t)$ for each fixed t, i.e., $\|\xi_\varepsilon(x^q - a)(t)\|_p$. This is due to the different topological structure induced by the norm in the space X_p. The comparison between (14) and (15) suggests that both algorithms may obtain different solution trajectories.

Example 4.1 In order to illustrate the use of the algorithm we present an application to generalized regression, according to the description given in (1). Assume that we observe three phenomena in the interval [0, 1], represented by the three functions

$$a_1(t) = (1 + \sin(t), 2 - t^2),$$
$$a_2(t) = (3 - 2t, 4 + t^4),$$
$$a_3(t) = (-1 + 2t^2, -2 - e^{3t}),$$

in $L^2[0, 1]$. The goal is to find another function that best fits the three given functions in the sense of the norm of the space $L^2[0, 1]$. For the ease of presentation we have chosen a weighted sum of the deviations as the measure of fitness function

$$f_0(x) = 0.25\|x - a_1\|_2 + 0.35\|x - a_2\|_2 + 0.4\|x - a_3\|_2.$$

Obviously, any other function under our hypothesis may have been chosen.

For the implementation of the algorithm we use Mathematica. We take as starting function $x_0(t) = (2 + 0.5t, 2 + 0.5t)$ and a sequence of $\varepsilon_n = \frac{0.01}{n}$. The iterations of the algorithm are shown in Table 1. In this table we represent $(q, (x_1^q(t), x_2^q(t)), f_0(x_1^q, x_2^q))$, i.e., the first column is the iteration number, the second column shows the expressions of the iterates and the last column is the objective value at the iteration. The convergence is achieved in few iterations up to an accuracy of 10^{-3}. The limit function is

$$(1.00907 - 0.01177t + 0.0027t^2 + 0.99276 \sin[t], 2.00637$$
$$-0.00135e^t - 0.99276t^2 + 0.00589t^4).$$

Our next example also illustrates the use of the algorithm. In addition, it shows that in order to solve Problem (2), the resolution of the pointwise version of the problem in each value of the interval I is not enough, because the optimal solution of (2) could not be attained in this way. This example proves that our approach is necessary to solve location problems with moving service facilities.

Example 4.2 Let us consider for $m = 2$ the space $X_{\frac{3}{2}} = L^{\frac{3}{2}}([0, 5], \mathbb{R}^2)$. In this space, we consider the demand functions

$$a_1(t) = (0, 0)\ \chi_{[0,2]}(t) + (5, 4)\ \chi_{[2,5]}(t)$$
$$a_2(t) = (4, 0)\ \chi_{[0,2]}(t) + (1, 2)\ \chi_{[2,5]}(t)$$
$$a_3(t) = (2, 4)\ \chi_{[0,2]}(t) + (7, 3)\ \chi_{[2,5]}(t),$$

where χ_I is the indicator function defined in (3).

Table 1 Iterations of the algorithm in the Example 4.1

1	$(1.04529 - 0.05880t + 0.01351t^2 + 0.96384\sin[t], 2.03177 - 0.00676e^t - 0.96384t^2 + 0.02940t^4)$	5.49275
2	$(1.03493 - 0.04530t + 0.01038t^2 + 0.97216\sin[t], 2.02455 - 0.00519e^t - 0.97216t^2 + 0.02265t^4)$	5.49191
3	$(1.02842 - 0.03688t + 0.00845t^2 + 0.97733\sin[t], 2.01997 - 0.00423e^t - 0.97733t^2 + 0.01844t^4)$	5.49137
4	$(1.02449 - 0.03179t + 0.00729t^2 + 0.98046\sin[t], 2.01720 - 0.00364e^t - 0.98046t^2 + 0.01589t^4)$	5.49104
5	$(1.02186 - 0.02837t + 0.00651t^2 + 0.98256\sin[t], 2.01535 - 0.00325e^t - 0.98256t^2 + 0.01418t^4)$	5.49082
6	$(1.01993 - 0.02587t + 0.00594t^2 + 0.9841\sin[t], 2.014 - 0.00297e^t - 0.9841t^2 + 0.01293t^4)$	5.49067
7	$(1.01844 - 0.02394t + 0.0055t^2 + 0.98528\sin[t], 2.01295 - 0.00274e^t - 0.98528t^2 + 0.01197t^4)$	5.49054
8	$(1.01725 - 0.02239t + 0.00514t^2 + 0.98624\sin[t], 2.01211 - 0.00257e^t - 0.98624t^2 + 0.01119t^4)$	5.49045
9	$(1.01626 - 0.02111t + 0.00484t^2 + 0.98702\sin[t], 2.01142 - 0.00242e^t - 0.98702t^2 + 0.01055t^4)$	5.49037
10	$(1.01543 - 0.02002t + 0.0046t^2 + 0.98769\sin[t], 2.01083 - 0.0023e^t - 0.98769t^2 + 0.01001t^4)$	5.4903
11	$(1.01471 - 0.01909t + 0.00438t^2 + 0.98826\sin[t], 2.01033 - 0.00219e^t - 0.98826t^2 + 0.00955t^4)$	5.49024
12	$(1.01408 - 0.01828t + 0.0042t^2 + 0.98876\sin[t], 2.00989 - 0.0021e^t - 0.98876t^2 + 0.00914t^4)$	5.49019
13	$(1.01353 - 0.01756t + 0.00403t^2 + 0.9892\sin[t], 2.0095 - 0.00202e^t - 0.9892t^2 + 0.00878t^4)$	5.49014
14	$(1.01304 - 0.01693t + 0.00389t^2 + 0.98959\sin[t], 2.00915 - 0.00194e^t - 0.98959t^2 + 0.00846t^4)$	5.4901
15	$(1.0126 - 0.01635t + 0.00375t^2 + 0.98994\sin[t], 2.00884 - 0.0019e^t - 0.98994t^2 + 0.00818t^4)$	5.49007
16	$(1.0122 - 0.01584t + 0.00364t^2 + 0.99026\sin[t], 2.00856 - 0.00182e^t - 0.99026t^2 + 0.00792t^4)$	5.49003
17	$(1.01184 - 0.015366t + 0.00353t^2 + 0.99055\sin[t], 2.00831 - 0.00176e^t - 0.99055t^2 + 0.00768t^4)$	5.49
18	$(1.0115 - 0.01493t + 0.00343t^2 + 0.99082\sin[t], 2.00807 - 0.00171e^t - 0.99082t^2 + 0.00747t^4)$	5.48998
19	$(1.0112 - 0.01454t + 0.00334t^2 + 0.99106\sin[t], 2.00786 - 0.00167e^t - 0.99106t^2 + 0.00727t^4)$	5.48995
20	$(1.01092 - 0.01417t + 0.00325t^2 + 0.99129\sin[t], 2.00766 - 0.00163e^t - 0.99129t^2 + 0.00708t^4)$	5.48993
21	$(1.01065 - 0.01383t + 0.00318t^2 + 0.9915\sin[t], 2.00748 - 0.00159e^t - 0.9915t^2 + 0.00691t^4)$	5.48991
22	$(1.01041 - 0.01351t + 0.0031t^2 + 0.99169\sin[t], 2.00731 - 0.00155e^t - 0.99169t^2 + 0.00676t^4)$	5.48989
23	$(1.01018 - 0.01322t + 0.00303t^2 + 0.99187\sin[t], 2.00715 - 0.00152e^t - 0.99187t^2 + 0.00661t^4)$	5.48987
24	$(1.00997 - 0.01294t + 0.00297t^2 + 0.99204\sin[t], 2.007 - 0.00148e^t - 0.99204t^2 + 0.0065t^4)$	5.48985
25	$(1.00977 - 0.01268t + 0.00291t^2 + 0.9922\sin[t], 2.00685 - 0.001456e^t - 0.9922t^2 + 0.00634t^4)$	5.48984
26	$(1.00958 - 0.01243t + 0.00285t^2 + 0.99235\sin[t], 2.00672 - 0.00143e^t - 0.99235t^2 + 0.00622t^4)$	5.48982
27	$(1.0094 - 0.0122t + 0.0028t^2 + 0.9925\sin[t], 2.0066 - 0.0014e^t - 0.9925t^2 + 0.0061t^4)$	5.4898
28	$(1.00923 - 0.01198t + 0.00275t^2 + 0.99263\sin[t], 2.00648 - 0.00138e^t - 0.99263t^2 + 0.00599t^4)$	5.48979
29	$(1.00907 - 0.01177t + 0.0027t^2 + 0.99276\sin[t], 2.00637 - 0.00135e^t - 0.99276t^2 + 0.00589t^4)$	5.48978

Within this framework we choose the globalising function

$$f_0(x) = \sum_{i=1}^{3} \omega_i ||x - a_i||_{\frac{3}{2}}$$

with weights $\omega_1 = \omega_2 = \dfrac{2}{5}$, and $\omega_3 = \frac{1}{5}$.

In order to solve this example we use the algorithm presented in Section 3 with the sequence $\varepsilon_n = \frac{0.01}{n}$, $\forall n \in \mathbb{N}$; and starting function

$$x^o(t) = (2, 0.5)\ \chi_{[0,2]}(t) + (4, 3.5)\ \chi_{[2,5]}(t).$$

The algorithm has been implemented in Mathematica and it stops after 25 iterations with an accuracy of 10^{-5}. Table 2 shows the iterations of the algorithm. The column *It.* gives the number of iterations; *Functions* gives the iterates and *Objective* the objective value of the problem for the corresponding iteration.

Note that for this example an optimal solution is

$$(1.18621, 0.127739)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t),$$

and the optimal objective value is 7.29059.

On the other hand, we also solve the problem pointwisely. This is to say, we solve the problem using the hyperbolic Weiszfeld algorithm applied to every point in the interval [0, 5]. Since we are considering demand functions with only two

Table 2 Iterations of Weiszfeld dynamic hyperbolic algorithm

It.	Functions	Objective
1	$(1.91775, 0.286981)\ \chi_{[0,2]}(t) + (4.21982, 3.32866)\ \chi_{[2,5]}(t)$	7.36395
2	$(1.69092, 0.182902)\ \chi_{[0,2]}(t) + (4.42443, 3.23562)\ \chi_{[2,5]}(t)$	7.31494
3	$(1.48868, 0.153807)\ \chi_{[0,2]}(t) + (4.55146, 3.2665)\ \chi_{[2,5]}(t)$	7.299
4	$(1.35367, 0.142395)\ \chi_{[0,2]}(t) + (4.62273, 3.32386)\ \chi_{[2,5]}(t)$	7.2931
5	$(1.27023, 0.135361)\ \chi_{[0,2]}(t) + (4.66099, 3.36491)\ \chi_{[2,5]}(t)$	7.29121
6	$(1.224, 0.131231)\ \chi_{[0,2]}(t) + (4.68044, 3.388)\ \chi_{[2,5]}(t)$	7.29072
7	$(1.20135, 0.129153)\ \chi_{[0,2]}(t) + (4.68957, 3.39932)\ \chi_{[2,5]}(t)$	7.29061
8	$(1.19159, 0.128246)\ \chi_{[0,2]}(t) + (4.69344, 3.4042)\ \chi_{[2,5])}(t)$	7.2906
9	$(1.1879, 0.127901)\ \chi_{[0,2]}(t) + (4.69489, 3.40605)\ \chi_{[2,5]}(t)$	7.29059
10	$(1.18668, 0.127787)\ \chi_{[0,2]}(t) + (4.69536, 3.40666)\ \chi_{[2,5]}(t)$	7.29059
11	$(1.18633, 0.127753)\ \chi_{[0,2]}(t) + (4.6955, 3.40684)\ \chi_{[2,5]}(t)$	7.29059
12	$(1.18624, 0.127744)\ \chi_{[0,2]}(t) + (4.69554, 3.40688)\ \chi_{[2,5]}(t)$	7.29059
13	$(1.18622, 0.127742)\ \chi_{[0,2]}(t) + (4.69555, 3.40689)\ \chi_{[2,5]}(t)$	7.29059
14	$(1.18621, 0.127741)\ \chi_{[0,2]}(t) + (4.69555, 3.40689)\ \chi_{[2,5]}(t)$	7.29059
15	$(1.18621, 0.127741)\ \chi_{[0,2]}(t) + (4.69555, 3.40689)\ \chi_{[2,5]}(t)$	7.29059
16	$(1.18621, 0.12774)\ \chi_{[0,2]}(t) + (4.69555, 3.40689)\ \chi_{[2,5]}(t)$	7.29059
17	$(1.18621, 0.12774)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t)$	7.29059
18	$(1.18621, 0.12774)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t)$	7.29059
19	$(1.18621, 0.12774)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t)$	7.29059
20	$(1.18621, 0.12774)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t)$	7.29059
21	$(1.18621, 0.12774)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t)$	7.29059
22	$(1.18621, 0.12774)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t)$	7.29059
23	$(1.18621, 0.127739)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t)$	7.29059
24	$(1.18621, 0.127739)\ \chi_{[0,2]}(t) + (4.69555, 3.4069)\ \chi_{[2,5]}(t)$	7.29059

Table 3 Pointwise iterations of Weiszfeld algorithm for the points in [0, 2]

It.	Functions	Objective
1	(2., 0.276606)	2.39908
2	(2., 0.158428)	2.39201
3	(2., 0.13055)	2.39163
4	(2., 0.126633)	2.39162
5	(2., 0.126332)	2.39162
6	(2., 0.12632)	2.39162
7	(2., 0.126319)	2.39162
8	(2., 0.126319)	2.39162

different steps, this is equivalent to solve two different classical Weber problems. The first one having demand points $(0, 0)$, $(4, 0)$ and $(2, 4)$ and the second one $(5, 4)$, $(1, 2)$ and $(7, 3)$. Using as starting points $(2, 0.5)$ and $(4, 3.5)$ respectively, Tables 3 and 4 show the iterations of these two problems.

The solutions obtained after the application of this procedure are $(2, 0.126319)$ for the problem in the interval [0, 2] and $(5, 4)$ for the problem in the interval [2, 5].

Table 4 Pointwise iterations of Weiszfeld algorithm for the points in [2, 5]

It.	Functions	Objective
1	(4.35746, 3.40509)	2.51804
2	(4.70544, 3.43239)	2.47524
3	(4.85959, 3.56313)	2.46
4	(4.91596, 3.6859)	2.45371
5	(4.94331, 3.77313)	2.45077
6	(4.95962, 3.83259)	2.4493
7	(4.97022, 3.87375)	2.44853
8	(4.97745, 3.90298)	2.44809
9	(4.98258, 3.92428)	2.44783
10	(4.98634, 3.94018)	2.44766
11	(4.98917, 3.95228)	2.44755
12	(4.99133, 3.96165)	2.44748
13	(4.99302, 3.96901)	2.44742
14	(4.99435, 3.97487)	2.44739
15	(4.99542, 3.97957)	2.44736
16	(4.99628, 3.98337)	2.44734
17	(4.99698, 3.98647)	2.44732
18	(4.99754, 3.989)	2.447318
19	(4.99801, 3.99107)	2.4473
20	(4.99839, 3.99278)	2.44729
21	(4.9987, 3.99417)	2.44728
22	(4.99896, 3.99532)	2.44728
23	(4.99917, 3.99626)	2.44728
24	(4.99934, 3.99703)	2.44727
25	(4.99948, 3.99765)	2.44727
26	(4.99959, 3.99815)	2.44727
27	(4.99968, 3.99856)	2.44727
28	(4.99975, 3.99888)	2.44727
29	(4.99981, 3.99914)	2.44726
30	(4.99999, 3.99999)	2.44726

Therefore, the solution to the problem using this approach is $(2, 0.126319)\chi_{[0,2]} + (5, 4)\chi_{[2,5]}$ and the objective value evaluated at this function is 7.61098.

The comparison of this value with 7.29059 (the objective value of the previously obtained solution) demonstrates that the pointwise application of the classical hyperbolic algorithm is not a substitute for the application of our algorithm.

5 Final remarks

The dynamic approach to single facility location problems is not new and can be seen as a natural way to improve the modeling of real world situations where demand is time dependent as for instance situations with seasonal demand.

In a previous paper, in 1999, we dealt with a dynamic formulation of the Weber problem on L^p spaces and showed that a modification of the Weiszfeld algorithm (1937) converges in the strong topology for each $p \in [1, 2]$. In this paper, we extends the above mentioned problem considering the minimization of a general increasing function g rather than the sum function. Our main result is the development of an algorithm based on a perturbation of a fixed point equation for which we prove global convergence to the optimal solution of the considered problem.

To get the global convergence for the perturbed algorithm, we impose that the demand functions and the starting iterate of the algorithm belong to a certain family of subspaces of L^p, called Sobolev spaces. However, although this condition seems to be a restriction most of the cases that can be considered are covered by this hypothesis. This is because Sobolev spaces are spaces of regular measurable functions which are the usual ones for representing trajectories. Moreover, this methodology has another important feature. Since the functions of the perturbed problems are differentiable, successive iterations of our algorithm can coincide (totally o partially) with a demand function. Such coincidence had to be avoided to prove the convergence in the previous models (Brimberg and Love 1993; Puerto and Rodríguez-Chía 1999).

Finally, it is very important to remark that the paper also proves that the optimal dynamic solution is not just the static solution taken over time. Example 4.2 clearly shows this counterintuitive result. Therefore, the considered dynamic single facility location problem, is worthwhile because it leads us to new results not being extensions of the static problem.

Further extensions of the material developed in this paper are possible in several lines considering different aspect of location analysis. Specifically, multifacility as well as conditional location problems with moving service facilites are natural extensions of the results in this paper. These two topics are currently under research and may be the content of a follow up paper.

Appendix

In this section we introduce some mathematical remarks needed for this paper. We start defining the so called Sobolev spaces $W^{1,p}$ which are subspaces of the L^p spaces of functions.

Definition 5.1 *The Sobolev space $W^{1,p}(I, \mathbb{R}^m)$ is the set*

$$W^{1,p}(I, \mathbb{R}^m) = \{x \in X_p \ : \ \exists g \in X_p \text{ such that } \int_I x(t)\phi'(t)\, dt$$
$$= -\int_I g(t)\phi(t)\, dt \quad \forall \phi \in C_c^1(I, \mathbb{R}^m)\}$$

where $C_c^1(I, \mathbb{R}^m)$ is the space of functions continuously differentiable with compact support. We denote $g = x'$, because if x is differentiable and its derivative belongs to X_p then the function g is its derivative.

Recall that $W^{1,p}(I, \mathbb{R}^m)$ is a Banach space with the norm defined as

$$||u||_{1,p} = ||u||_p + ||u'||_p.$$

In order to improve the readability of the paper we include without proof several properties which hold in these spaces and which are used to prove the strong convergence results. The proofs of these properties and further details on Sobolev spaces can be found in the book of Brezis (1983).

Lemma 5.1 *The following assertions hold*

(1) *Let $u, v \in W^{1,p}(I, \mathbb{R}^m)$ then $uv \in W^{1,p}(I, \mathbb{R}^m)$*
(2) *There exists a compact imbedding from $W^{1,p}(I, \mathbb{R}^m)$ into X_p.*

The existence of a compact imbedding is a very important fact because it implies that if a sequence converges in the weak topology of $W^{1,p}(I, \mathbb{R}^m)$ then it also converges in the strong topology of X_p.

Finally, we recall some concepts concerning different modes of convergence on normed spaces which will be used in the paper. Let X_p be a normed space equipped with the norm $|| \cdot ||$ and denote by X_p^* its algebraic dual with the pairing between $x \in X_p$ and $z \in X_p^*$ given by

$$\langle z, x \rangle = \int_I x(t)z(t)\, dt. \tag{16}$$

Remark that $X_p^* = X_q$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

Definition 5.2 *A sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ is said to be strongly convergent to $\bar{x} \in X$ if*

$$\lim_{n \to \infty} ||x_n - \bar{x}|| = 0.$$

In the same way, $\{x_n\}_{n \in \mathbb{N}}$ is said to be weakly convergent to $\bar{x} \in X$ if for all $z \in X^*$

$$\lim_{n \to \infty} \langle z, x_n - \bar{x} \rangle = 0.$$

It is well-known that the strong convergence always implies the weak one but in general the converse does not hold.

Another kind of convergence is the so called epi-convergence. The epi-convergence is very important because it states relationships between the convergence of functionals and the convergence of the sequence of their minima. For the sake of completeness we recall Definition 1.9 in the book of Attouch (1984). Let $\{g\,;\;g^v\;v=1,\dots\}$ be a collection of extended-values functions. We say that g^v epi-converges to g, if for all x

$$\inf_{x^v\to x}\lim\inf_{v\to\infty} g^v(x^v)\geq g(x)$$
$$\inf_{x^v\to x}\lim\sup_{v\to\infty} g^v(x^v)\leq g(x)$$

where the infima are taken with respect to all the sequences converging to x.

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